Calculation of Mutual Information for Nonlinear Optical Fiber Communication Channel at Large SNR within Path-Integral Formalism

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In our work calculate analytically the mutual information $I_{P[X]}$ [1,2] (difference between entropy $H[Y]$ of the output signal and conditional entropy $H[Y|X]$) for the fibre optical channel described by the nonlinear Schrödinger equation (NLSE) with additive white Gaussian noise in the leading nonzero order in nonlinearity and at large signal-to-noise ration (SNR):

$$\partial_z\psi + i\beta\partial_t^2\psi - i\gamma|\psi|^2\psi = \eta(z,t)\,, \qquad (1)$$

where $\psi(z,t)$ is the outcoming signal, γ is the Kerr nonlinerity, $\beta = \beta_2/2$ is the dispersion parameter, $\eta(z,t)$ is the white Gaussian noise: $\langle\eta(z,t)\bar{\eta}(z',t')\rangle_\eta = Q\delta(z-z')\delta(t-t')$, where Q is a power of the white Gaussian noise (per unit length and frequency). The NLSE is one of the fundamental models in nonlinear physics and has a broad range of applications, including fibre-optic communications – the backbone of the Internet.

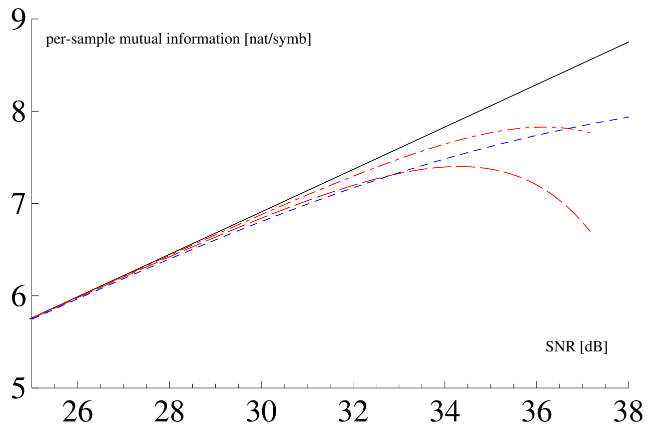

Figure 1: The per-sample mutual information $i_{P[X]} = I_{P_G[X]}/M$ (in nat/symb) as the function of SNR (in dB). The solid black line, red long-dashed line, blue dashed, red dashed-dotted line correspond to $i_{P[X]}$ for a linear channel, channel with dispersion $\widetilde{\beta} = 200$, nondispersive channel, and the channel with $\widetilde{\beta} = 800$, respectively

In our approach we use the representation of the mutual information in the form of path-integral

$$I_{P[X]} = \int \mathcal{D}X\mathcal{D}Y\, P[X]P[Y|X]\log\frac{P[Y|X]}{P_{out}[Y]} = H[Y] - H[Y|X] \qquad (2)$$

where $P[X]$ is the probability density function (PDF) of the initial signal X with the fixed finite average power P_{ave}. The function $P[Y|X]$ here is the conditional probability density function, that is the probability density of receiving output signal Y when the input signal is X. The output signal PDF $P_{out}[Y] = \int\mathcal{D}X\,P[X]P[Y|X]$. The characteristic feature of our approach is that we use the formulation for the conditional PDF $P[Y|X]$ through the path-integral (Martin-Siggia-Rose formalism) [3]. In the frequency domain it reads

$$P[Y(\omega)|X(\omega)] = \Lambda\, e^{-S[\Psi_\omega(z)]/Q}, \qquad (3)$$

$$\Lambda = \int\limits_{\phi_\omega(0)=0}^{\phi_\omega(L)=0} \mathcal{D}\phi\, e^{-\{S[\Psi_\omega(z)+\phi_\omega(z)]-S[\Psi_\omega(z)]\}/Q}\,. \qquad (4)$$

Here the functional $S[\psi]$ is the action which has the form

$$S[\psi] = \int\limits_0^L dz\int\frac{d\omega}{2\pi}\Big|\partial_z\psi_\omega(z) - i\beta\omega^2\psi_\omega(z) - i\gamma\int\frac{d\omega_1 d\omega_2}{(2\pi)^2}\psi_{\omega_1}(z)\psi_{\omega_2}(z)\bar{\psi}_{\omega_3}(z)\Big|^2, \qquad (5)$$

where $\omega_3 = \omega_1 + \omega_2 - \omega$. The function $\Psi_\omega(z)$ in Eq. (3) is referred to as the "classical trajectory". It is the extremum function of the action S, i.e. the action variation is equal to zero on the function $\Psi_\omega(z)$: $\delta S[\Psi] = 0$ with the boundary conditions $\Psi_\omega(0) = X(\omega)$, $\Psi_\omega(L) = Y(\omega)$. For our calculations we use the Gaussian input signal PDF $P[X] = P_G[X]$ with the average power $P \gg QL$ ($SNR = P/(QL) \gg 1$).

In our work we obtain the general expression for the entropies ($H[Y]$ and $H[Y|X]$) and for the mutual information $I_{P[X]}$ through the path-integral which is convenient both for perturbative (small nonlinearity case) and non-perturbative analysis.

For the small nonlinearity case we demonstrate that the first nonlinear correction to the mutual information for the channel with dispersion is negative and quadratic in the nonlinearity parameter γ:

$$I_{P_G[X]} = M \log \mathrm{SNR} - M\frac{\widetilde{\gamma}^2}{3} g(\widetilde{\beta}) + \mathcal{O}(\widetilde{\gamma}^4), \tag{6}$$

where M is the number of complex meaning channels, $\widetilde{\gamma} = \gamma PLW/(2\pi)$ is the dimensionless nonlinearity parameter (P is the signal power per unit frequency, L is the length of the channel, W is the frequency bandwidth), and we have introduced the function g of dimensionless dispersion parameter $\widetilde{\beta} = \beta LW^2$ that reads

$$g(\widetilde{\beta}) = 4! \sum_{n=0}^{\infty} \frac{(-1)^n \widetilde{\beta}^{2n} \left[(4n+2)! + (1+2n)!^2\right]}{2^{2n-1}(2n+4)!(4n+3)!(1+2n)^2}. \tag{7}$$

For large $\widetilde{\beta} = \beta LW^2$ one has

$$g(\widetilde{\beta}) \sim \frac{16\pi}{\widetilde{\beta}} \left(\log\frac{\widetilde{\beta}}{2} + \gamma_E - \frac{23}{6} \right) + \mathcal{O}\left(\widetilde{\beta}^{-3/2}\right), \tag{8}$$

where $\gamma_E \approx 0.577$ is the Euler constant. There is the indication that at large $\widetilde{\beta}$, the effective parameter of the perturbative series is $\tilde{\gamma}^2 \log(\widetilde{\beta})/\widetilde{\beta}$ rather than $\tilde{\gamma}^2$.

In our work we compare the result (6) for the mutual information of the channel with nonzero dispersion and the exact result [4] for the nonlinear nondispersive channel:

$$I^{(\beta=0)}_{P_G[X]} = M \log \mathrm{SNR} - \frac{M}{2} \int_0^{\infty} d\tau e^{-\tau} \log\left(1 + \frac{\tau^2 \tilde{\gamma}^2}{3}\right). \tag{9}$$

We show that there is the region of the parameter SNR where the obtained mutual information is greater than that obtained for the channel with the zero dispersion. We also show that the region becomes wider with increasing of the dispersion parameter: see the Fig. For the example in the Fig. we choose following parameters: $\beta = 20\,\mathrm{ps}^2/\mathrm{km}$, $L = 1000\,\mathrm{km}$, $\gamma = 1.31(\mathrm{Wkm})^{-1}$, $W = 100\,\mathrm{GHz}$, $P_{noise} = QLW/2\pi = 5.3 \times 10^{-4}\mathrm{mW}$.

References

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